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Valuation of Linear Interest Rate Derivatives: Progressing from Single- to Multi-Curve Bootstrapping

White Paper

Patrick Brugger

Dr. habil. Jörg Wenzel

Department Financial Mathematics
Fraunhofer Institute for Industrial Mathematics ITWM

Contact:
Dr. habil. Jörg Wenzel
joerg.wenzel@itwm.fraunhofer.de

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Fraunhofer ITWM
Fraunhofer-Platz 1
67663 Kaiserslautern
Germany

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1 Description

At the base of each financial market lies the valuation of its instruments. Until the financial crisis of 2007–2008, the single-curve approach and its bootstrapping technique were used to value linear interest rate derivatives. Due to lessons learnt during the crisis, the valuation process for derivatives has been fundamentally changed. The aim of this white paper is to explain the foundation of the single-curve approach, why and how the methodology has been changed to the multi-curve approach and how to handle the computations if the collateral is being held in another currency. Moreover, we provide a brief overview of how interest rate models, which are used to price non-linear interest rate derivatives, have been extended after the crisis. Finally, we take a look at the history and future of Libor, which will be phased out by the end of 2021.

In particular, this document is an excellent starting point for someone who has had little or no prior exposure to this very relevant topic. At the time of publication, we were not aware of any similarly comprehensive resource. We intend to fill this gap by explaining in detail the fundamental concepts and by providing valuable background information. The last chapter about Libor can be studied independently from the remaining parts of this document.

2 Introduction: Interest Rate Derivatives, Libor and Zero-Bond Curves

»**Financial market**« is a generic term for markets where financial instruments and commodities are traded. **Financial instruments** are monetary contracts between two or more parties. A **derivative** is a financial instrument that *derives* its value from the performance of one or more underlying entities. For instance, this set of entities can consist of assets (such as stocks, bonds or commodities), indexes or interest rates and is itself called »underlying«. Derivatives are either traded on an **exchange**, a centralised market where transactions are standardised and regulated, or on an **over-the-counter (OTC) market**, a decentralised market where transactions are not standardised and less regulated. OTC markets are sometimes also called »off-exchange markets«. The OTC derivatives market grew exponentially from 1980 through 2000 and is now the largest market for derivatives. The gross market value of outstanding OTC derivatives contracts was USD 15 trillon in 2016, which corresponds to one fifth of that year's gross world product, see [3] and [36]. As we will learn later in Section 4.1.1, additional regulations were imposed upon the OTC market due to its role during the financial crisis of 2007–2008.

We are especially interested in valuing derivatives whose underlying is an interest rate or a set of different interest rates and call these derivatives **interest rate derivatives (IRDs)**. One of the most important forms of risk that financial market participants face is interest rate risk. This risk can be reduced and even eliminated entirely with the help of IRDs. Furthermore, IRDs are also used to speculate on the movement of interest rates and are mainly traded OTC. Around 67% of the global OTC derivatives market value arises from OTC traded IRDs, see [3]. IRDs can be divided into two subclasses:

- **Linear IRDs** are those whose payoff is linearly related to their underlying interest rate. Examples of this class are forward rate agreements, futures and interest rate swaps.[1]

  Until the financial crisis, the single-curve approach was used to price these IRDs. In Chapter 3, we will describe how it works, what the distinctive assumptions are and why they are flawed. Afterwards, in Chapter 4, we will discuss the multi-curve approach, which is now market standard for pricing linear IRDs. We will then have a brief look at the multi-currency case in Chapter 5.

- **Non-linear IRDs** are all the remaining instruments, i.e. those whose payoff evolves non-linearly with the value of the underlying. Basic examples are caps, floors and swaptions. However, this family of IRDs is very large and also includes very complex derivatives, such as autocaps, Bermudan swaptions, constant maturity swaps and zero coupon swaptions.

  We need interest rate models to price such IRDs, but they are not the focus of this paper. Nevertheless, in Chapter 6 we will provide a brief summary of how existing families of models were extended to the new framework due to the deeper market understanding and name the most relevant publications. Additionally, we point out links to results that were developed earlier in this paper.

> **Aim:** We want to **construct interest rate curves** that enable us to price any linear IRD of interest. For this purpose, we will use prices quoted on the market as input factors to a technique called »bootstrapping«.

Remark. The **London Interbank Offered Rate (Libor)** is the trimmed average of interest rates estimated by each of the leading banks in London that they would be charged

[1] Forward rate agreements and interest rate swaps will play a crucial role in this white paper and will be introduced in more detail later on.

if they had to borrow unsecured funds in reasonable market size from one another. Currently, it is calculated daily for five currencies (CHF, EUR, GBP, JPY, USD), each having seven different tenors (1d, 1w, 1m, 2m, 3m, 6m, 12m). Hence, there exist in total 35 distinct Libor rates. We omit the specification of the currency when not needed and write Δ-Libor for the Libor with tenor Δ. The banks contributing to Libor belong to the upper part of the banks in terms of credit standing and were considered virtually risk-free prior to the crisis. See Chapter 7 for more background information and for an explanation of why Libor will be phased out by the end of 2021.

Remark. Similar reference rates set by the private sector are, for example, the Euro Interbank Offered Rate (Euribor), the Singapore Interbank Offered Rate (Sibor) and the Tokyo Interbank Offered Rate (Tibor). Everything we do is applicable to all interbank offered rates. We refer solely to Libor due to the better readability and since this is an established standard in the literature.

In the following, we assume that the considered IRDs only have one underlying interest rate and that this rate is always Libor. Before we go into further details, we introduce some basic definitions:

Definition 1. We consider a **stochastic short rate model** and denote the **short rate** by $r(s)$ at time point s. This rate is the continuously compounded and annualised interest rate at which a market participant can borrow money for an infinitesimally short period of time at s. The (stochastic) **discount factor** $D(t,T)$ between two time instants t and T is the amount of money at time t that is »equivalent« to one unit of money payable at time T and is given by

$$D(t,T) \coloneqq \exp\left(-\int_t^T r(s)\mathrm{d}s\right).$$

Just like exchange rates can be used to convert cash being held in different currencies into one single currency, discount factors can be interpreted as special exchange rates which convert cash flows that are received across time into another »single currency«, namely into the present value of these future cash flows. Let us assume we know today ($t=0$) that we will receive X units of money in one year ($T=1$). The present value of this future cash flow can then be calculated as $D(0,1)\cdot X$. For obtaining the present value in the case that we have several future cash flows, we just take the sum of the respective present values.

Definition 2. A **zero-coupon bond** with maturity T (T-bond) is a riskless contract that guarantees its holder the payment of one unit of money at time T with no intermediate payments. The contract value at time $t \leq T$ is denoted by $P(t,T)$. Zero-coupon bonds are sometimes also called »discount bonds«.

Note that $P(T,T)=1$ and if interest rates are positive we have for $t \leq T \leq T'$

$$P(t,T) \geq P(t,T')\ .$$

For the valuation of linear IRDs we will need the prices of different zero-coupon bonds, since it can be shown that

$$P(t,T) = \mathbb{E}_t[D(t,T)]\ , \tag{1}$$

where the expectation is taken with respect to the risk neutral pricing measure and the filtration $\mathcal{F}_t$, which encodes the market information available up to time t, see also Section 4.1.2. Due to this crucial relation of discount factors and zero-coupon bonds, we sometimes use expressions such as »we discount with $P(t,T)$«. In conclusion, we are interested in the following curve:

Definition 3. The **zero-bond curve** at time t, with $t \leq T$, is given by the mapping

$$T \mapsto P(t,T)\ .$$

This curve is sometimes also called »discount curve« or »term structure curve«.

Using the **bootstrapping** technique, which will be described in the next chapter, we first obtain a finite number of the zero-bond curve's values from some given input data. Roughly speaking, in a bootstrap calculation we determine a curve $C : T \mapsto C(T)$ iteratively, where we get the unknown point $C(T_i)$ at T_i by a calculation that depends on previous points of the curve, $\{C(T_j) : j < i\}$. Afterwards, we use this finite number of values to generate the rest of the curve via inter- and/or extrapolation techniques. So, essentially, »bootstrapping« refers to forward substitution in the context of zero-bond curve construction.

Remark. We want to stress that when we use the term »bootstrapping«, we do not refer to the statistical method, let alone to any of its many other meanings. Usually, bootstrapping refers to a self-starting process that is supposed to proceed without external input. It is, by the way, also the origin of the term »booting«, used for the process of starting a computer by loading the basic software into the memory which will then take care of loading other software as needed. Etymologically, the term appears to have originated in the early 19th-century United States, particularly in the phrase »pull oneself over a fence by one's bootstraps«, to mean an absurdly impossible action.

3 Single-Curve Approach: One Curve Is Not Enough

Assumption: All linear IRDs depend on only one zero-bond curve.

Procedure: With this single zero-bond curve we

1. calculate the **forward rates** with which we obtain the future cash flows and
2. **discount** these future cash flows

at the same time to price our linear IRD of interest.

For the construction of this curve it is allowed to use **any set of liquid linear IRDs** on the market with increasing maturities and which have Libor as an underlying. Liquid instruments are those with negligible bid-ask spread, which basically means that supply meets demand, so that they can be converted into cash quickly and easily for full market price. In particular, the allowed sets of **instruments do not have to be homogeneous**, i.e. they can have different Libor indexes as underlying, such as 1m-Libor, 3m-Libor ...

It is important to realise that:

- Until the financial crisis of 2007–2008, Libor was seen as a good proxy for the theoretical concept of the risk-free rate, which motivated its usage for discounting.
- The usage of the same curve to discount the cash flows is a modelling choice and not a contractual obligation and thus is theoretically open to debate.

As we will see, these two points are **crucial differences to the multi-curve approach**, which is the method recommended by leading experts in the field, see for instance [15], where Henrad first proposed a different approach in 2007, and also [6], [25] and [27].

Nevertheless, we first start with the single-curve approach as it not only deepens our general understanding by outlining the historic evolution, but also uses a bootstrapping technique that will be relevant for the multi-curve approach later on.

3.1 Single-Curve Bootstrapping

In the following, $\tau(t,T)$ denotes the time period in years between time point t and T according to a specific day count convention. We assume the Actual/360 one, as Libor for all currencies except GBP (there it is the Actual/365 one) is based on it and, in general, it is the most prevalent day count convention for money market instruments with maturity below one year. It is determined by the factor $\frac{1}{360}\cdot$Days(t, T), i.e. one year is assumed to consist of 360 days, see [29].

Before we illustrate the bootstrapping procedure, we would like to introduce the concept of a simply compounded spot rate $L(t,T)$. By an arbitrage argument[2], one unit of currency at time T should be worth $P(t,T)$ units of currency at time t, see (1). Hence, we want that the following equation holds:

$$1 = P(t,T)\Big(1+\tau(t,T)\cdot L(t,T)\Big), \tag{2}$$

[2] An arbitrage opportunity is the possibility to make a riskless profit in a financial market without net investment of capital. The no-arbitrage principle states that a mathematical model of a financial market should not allow any arbitrage possibilities.

Single-Curve Approach: One Curve Is Not Enough

i.e. we assume that $L(t,T)$ is a riskless lending rate. This leads us to the following definition:

Definition 4. The **simply compounded spot rate** at time t for maturity T is defined as

$$L(t,T) := \frac{1 - P(t,T)}{\tau(t,T) \cdot P(t,T)} \ . \tag{3}$$

Important simply compounded spot rates are the market Libor rates, which motivates the above notation $L(t,T)$.

Definition 5. In a **fixed for floating interest rate swap** two parties periodically exchange interest rate payments on a given notional amount N. One party pays a fixed rate whereas the other pays a floating rate. These instruments are particularly useful for reducing or eliminating the exposure to interest rate risk. According to the market conventions of a given currency the fixed payment schedule has standard periods, for instance one year for EUR or six months for USD, see [29]. We call the sum of fixed payments the **fixed leg** and the sum of the floating payments the **floating leg**.

Remark. In a general **interest rate swap (IRS)**, each of the two parties has to pay either a fixed or a floating interest rate on a given notional amount N to its counterparty. An IRS is traded OTC and its notional amounts are never exchanged, as the term »notional« suggests.[3] The most common form of an IRS is a fixed for floating swap and the least common form is a fixed for fixed swap. IRSs constitute with 60% the largest part of the global OTC derivatives gross market value in 2017, see [3]. Consequently, they also represent by far the largest part of all OTC traded IRDs.

Prior to the financial crisis of 2007–2008, we were, for example, provided with the rates of Fig. 1 for bootstrapping the discount factors. So, in this case, Libor rates $L(0,T)$, $T \in \{1m, 3m, 6m, 12m\}$ are used as input rates until 12 months and swap fixed rates $S(0,T)$, $T \in \{2y, 3y, \ldots, 30y, 40y, 50y\}$, from 2 years to 50 years. As mentioned previously, any set of liquid vanilla interest rate instruments on the market with increasing maturities could be used, e.g. in addition to the ones above also mid-term futures or forward rate agreements on 3m-Libor.

Index	Type	Duration
1	Libor rate	1 month
2	Libor rate	3 months
3	Libor rate	6 months
4	Libor rate	12 months
5	Swap fixed rate	2 years
6	Swap fixed rate	3 years
⋮	⋮	⋮
32	Swap fixed rate	29 years
33	Swap fixed rate	30 years
34	Swap fixed rate	40 years
35	Swap fixed rate	50 years

Fig. 1: Input rates for bootstrapping prior to the financial crisis

Clearly, the resulting bootstrapped curve is a curve, where the prices of the instruments used as an input to the curve coincide with the prices that are calculated using this curve when valuing these same instruments.

[3] Hence, it does not make too much sense to estimate the IRS market size and risk by adding up the notional values of all outstanding IRSs. Unfortunately, this is still a common practice and even used in regulatory calculations, see also [14].

Single-Curve Approach: One Curve Is Not Enough

Using the bootstrapping technique, we are now going to extract $N = 35$ distinct grid points of the zero-bond curve $P(0, \cdot)$ from the above N distinct rates. These grid points will be the values $P(0, T)$ with

$$T \in \{1m, 3m, 6m, 12m, 2y, 3y, \ldots, 30y, 40y, 50y\} \ .$$

To this end, different calculations apply – one for $T \leq 1y$, one for $1y < T \leq 30y$ and one for $30y < T$:

- For $T \leq 1y$:

  From equation (2) follows immediately that

  $$P(0, T) = \frac{1}{1 + \tau(0, T)L(0, T)} \ .$$

  Note that in the case of EUR and GBP Libor, the fixing date of Libor corresponds to its value date, i.e. the rate is set on the same day that the banks contribute their submissions. However, for all currencies other than EUR and GBP the value date will fall two London business days after the fixing date, see [29]. If this subsequent date is

  (a) not a market holiday, we have to consider the overnight rate r_{ON} for the first day and the spot next rate r_{SN} for the day after to get the exact bond prices, see [20]. So in this case we would get

  $$P(0, T) = \frac{1}{1 + \tau(0, 1d)r_{ON} + \tau(1d, 2d)r_{SN} + \tau(2d, T)L(0, T)} \ .$$

  (b) a market holiday, the value date will roll onto the next date which is a normal business day both in London and in the principal financial centre of the relevant currency.

- For $1y < T \leq 30y$:

  From one year onwards, we use swap fixed rates to calculate the bond prices. We consider swap rates that are quoted annually and where the fixed rate is paid annually, too, as is the case in the EUR market, see [29]. The starting point is again an equation: For instance, when $T = 2y$ we start with

  $$1 = S(0, 2y)\tau(0, 1y)P(0, 1y) + \Big(\tau(1y, 2y)S(0, 2y) + 1\Big)P(0, 2y) \ .$$

  This treats the swap as a 2-year $S(0, 2)$ fixed-rate bullet bond priced at par value of 1. A bullet bond is a debt instrument whose entire principal value is paid all at once on the maturity date, as opposed to amortizing the bond over its lifetime. It cannot be redeemed prior to maturity.

  All payments on this hypothetical bond prior to the maturity date – which is in this case only one payment after one year – are multiplied by the corresponding bond price. At maturity we discount the last payment and the repaid principal with the unknown factor $P(0, 2y)$. Hence, we obtain

  $$P(0, 2y) = \frac{1 - S(0, 2y)\tau(0, 1y)P(0, 1y)}{1 + \tau(1y, 2y)S(0, 2y)} \ .$$

  To get $P(0, 3y)$, we start from

  $$\begin{aligned} 1 = S(0, 3y)\tau(0, 1y)P(0, 1y) &+ S(0, 3y)\tau(1y, 2y)P(0, 2y) \\ &+ \Big(\tau(2y, 3y)S(0, 3y) + 1\Big)P(0, 3y) \ , \end{aligned}$$

and so on ... So, in general we have

$$P(0,T) = P(0,T_n) = \frac{1 - S(0,T) \cdot \sum_{i=1}^{n-1} \tau(T_{i-1},T_i) \cdot P(0,T_i)}{1 + S(0,T) \cdot \tau(T_{n-1},T_n)} \;, \tag{4}$$

where we write T_n to denote that the corresponding swap has a duration of n years and we put $T_0 := 0$. It is possible to solve the above equations for the different T at once using matrix computations, as illustrated in [9].

Once again, we have to pay attention to different market conventions: For example, in the USD market the fixed rate is payed semi-annually. However, we do not have, for instance, a 1y6m swap fixed rate to calculate $P(0, 1y6m)$. The simple solution is to interpolate the missing swap fixed rates with a suitable interpolation technique, see the next remark, and to apply a similar procedure as described for the next case, $30y < T$, in more detail. The used interpolated rates are also referred to as »synthetic rates« in the literature.

- For $30y < T$:

  The data for long durations is usually sparse. Therefore we use an iterative approach for calculating the bond prices, where we obtain a value $P_i(0,T)$ in each iteration. Let $T = 40$ and for $i = 1$ we set

  $$P_1(0,40) := \Big(\frac{1}{1 + S(0,40)}\Big)^{40}.$$

  We proceed as follows:

  1. Derive the bond prices for the years $31, \ldots, 39$ using log-linear interpolation, or any other suitable interpolation technique, see the next remark, between $P(0,30)$ and $P_i(0,40)$. We assume for now that the obtained values are the true ones.

  2. Calculate $P_{i+1}(0,40)$ as in equation (4) with the obtained values of the previous step.

  This routine is repeated until for the k-th repetition the obtained improvement is negligibly small, e.g.

  $$|P_{k+1}(0,40) - P_k(0,40)| < 10^{-8} \;.$$

  For the following years, the same routine is applied.

After obtaining the bond prices at the given time points we can interpolate between them to get the entire zero-bond curve.[4]

Remark. Usually, the interpolation is done on the logarithm of the bond prices using one or the other interpolation method. In [30], the usage of different interpolation methods for curve construction is discussed in detail. The authors provide some warning flags and stress that natural cubic splines possess the non-locality property, since it uses information from three time points. Non-locality means that if the input value at time point t_i is changed, the interval (t_{i-l}, t_{i+u}), where the values of the curve change, is rather large. The method choice is always subjective and needs to be decided on a case by case basis.

As explained previously, in the single-curve approach we only use the information of this *unique* zero-bond curve to price *any* linear IRD in a given currency.

[4] See Section 4.3.2 for an alternative, the so-called »best fit approach«, which is being used by most central banks.

3.2 The Single-Curve Approach in Light of the Financial Crisis

Single-Curve Approach: One Curve Is Not Enough

In the following, we illustrate in the first proof of Lemma 3.1 how the usage of this one curve works and provide a brief explanation why this procedure is not recommendable. This motivates the usage of the multi-curve approach, which will be covered in Chapter 4.

Definition 6.

- A **standard forward rate agreement (FRA)** is a contract involving three time instants: (i) the current time t, (ii) the expiry time T_{i-1} and (iii) the maturity time T_i, with $t \leq T_{i-1} < T_i$.
- The contract gives its holder an interest rate payment for the period between T_{i-1} and T_i at maturity T_i, which corresponds to the difference between the fixed rate $L(t, T_{i-1}, T_i)$ and the floating spot rate $L(T_{i-1}, T_i)$.
- $L(t, T_{i-1}, T_i)$ is the risk-free rate for the time interval $[T_{i-1}, T_i]$ determined at time t and we call it the **simply compounded forward rate** or the **FRA rate**.

Lemma 3.1

Using the single-curve approach we get

$$L(t, T_{i-1}, T_i) = \frac{1}{\tau(T_{i-1}, T_i)} \left(\frac{P(t, T_{i-1})}{P(t, T_i)} - 1 \right). \tag{5}$$

To show this we provide two proofs. The first one illustrates how we get the above expression and the second one uses a replication argument to which we will come back later.

Proof 1. Here we follow the proof of [7]. Let N denote the contract nominal value and K the simply compounded forward rate with expiry time T_{i-1} and maturity time T_i, i.e. $K = L(t, T_{i-1}, T_i)$. At time T_i one receives $\tau(T_{i-1}, T_i)KN$ units of currency and pays $\tau(T_{i-1}, T_i)L(T_{i-1}, T_i)N$. The payoff of the contract at time T_i is therefore

$$\begin{aligned} \text{FRA}(t, T_{i-1}, T_i, N) &= N\tau(T_{i-1}, T_i)\Big(K - L(T_{i-1}, T_i)\Big) \\ &\stackrel{(3)}{=} N\left(\tau(T_{i-1}, T_i)K - \frac{1}{P(T_{i-1}, T_i)} + 1\right). \end{aligned} \tag{6}$$

Because of the assumption that we are using the single-curve approach we can discount the cash flows with the same curve. Note that the amount $1/P(T_{i-1}, T_i)$ at time T_i is worth one unit of money at time T_{i-1}. One unit of money at time T_{i-1} is in turn equal to an amount of $P(t, T_{i-1})$ at time t. On the other hand, the amount $\tau(T_{i-1}, T_i)K + 1$ from (6) at time T_i is worth $P(t, T_i)\tau(T_{i-1}, T_i)K + P(t, T_i)$ at time t. The total value of the forward rate agreement at time t is

$$N\Big(P(t, T_i)\tau(T_{i-1}, T_i)K - P(t, T_{i-1}) + P(t, T_i)\Big).$$

If we equate this to zero for no-arbitrage reasons and solve for K we obtain the above statement. □

Proof 2. At first, we construct two separate strategies at different time points:

Time	FRA-Strategy	Bond-Strategy
t	• buy one FRA **cash flow** 0	• sell one T_{i-1}-bond • buy $\frac{P(t,T_{i-1})}{P(t,T_i)}$ T_i-bonds **cash flow** 0
T_{i-1}	• invest 1 at $L(T_{i-1},T_i)$ **cash flow** -1	• pay back T_{i-1}-bond **cash flow** -1
T_i	• receive from investment: $1 + L(T_{i-1},T_i) \cdot \tau(T_{i-1},T_i)$ • receive from FRA: $(L(t,T_{i-1},T_i) - L(T_{i-1},T_i)) \cdot \tau(T_{i-1},T_i)$ **cash flow** $1 + L(t,T_{i-1},T_i) \cdot \tau(T_{i-1},T_i)$	• receive from T_i-bonds: $\frac{P(t,T_{i-1})}{P(t,T_i)}$ **cash flow** $\frac{P(t,T_{i-1})}{P(t,T_i)}$

Due to the no-arbitrage assumption we get

$$\begin{aligned} 1 + L(t,T_{i-1},T_i) \cdot \tau(T_{i-1},T_i) &= \frac{P(t,T_{i-1})}{P(t,T_i)} \\ \Leftrightarrow \qquad L(t,T_{i-1},T_i) &= \frac{1}{\tau(T_{i-1},T_i)} \left(\frac{P(t,T_{i-1})}{P(t,T_i)} - 1 \right) . \end{aligned}$$

□

Remark. Using (3) and Lemma 3.1 it follows that the simply compounded spot rate is equal to the simply compounded forward rate if $t = T_{i-1}$, i.e.

$$L(t,T_i) = \frac{1 - P(t,T_i)}{\tau(t,T_i) \cdot P(t,T_i)} = L(t,t,T_i) \; . \tag{7}$$

In the literature, the simply compounded spot rate $L(t,T_i)$ is sometimes defined as the value of the simply compounded forward rate $L(t,T_{i-1},T_i)$ with $t = T_{i-1}$. However, we started with the simply compounded spot rate and derived the value of the simply compounded forward rate in Lemma 3.1 because the proofs provide us with a deeper understanding of the single-curve approach, as we will realise very soon.

At this point, we already introduce the concept of the net present value, which will be covered in more detail in Section 4.1.2:

Definition 7. We define the **net present value (NPV)** at time t of a financial transaction with net cash flows[5] Cash(T_i) at time points T_i, $i \in \{1,\dots,M\}$, by

$$\text{NPV}(t) := \sum_{i=1}^{M} P(t,T_i) \mathbb{E}_t^i[\text{Cash}(T_i)] \; , \tag{8}$$

where $\mathbb{E}_t^i[\cdot]$ denotes the expectation under the T_i-forward measure associated with the risk-free zero-coupon bond $P(t,\cdot)$ and the filtration $\mathcal{F}_t$.

Example 1. The NPV of a Libor swap's floating leg at time t is given by

$$\text{NPV}(t) = \sum_{i=1}^{M} P(t,T_i) L(t,T_{i-1},T_i) \tau(T_{i-1},T_i) \; ,$$

[5] »Net cash flow« refers to the difference between cash inflows and cash outflows in a given period of time. Accordingly, the »net present value« is the difference between the present value of cash inflows and the present value of cash outflows in a given period of time.

where T_M denotes the maturity date of the swap. Hence, it is given by simply summing up the values of the single floating payments discounted with the corresponding zero-coupon bond price, compare the paragraph after Definition 1.

Lemma 3.2

For the NPV of a Libor swap's floating leg at time t it holds

$$\text{NPV}(t) = 1 - P(t, T_M) \; . \tag{9}$$

Proof.

$$\begin{aligned}
\text{NPV}(t) &= \sum_{i=1}^{M} P(t,T_i) L(t,T_{i-1},T_i) \tau(T_{i-1},T_i) \\
&\overset{(5)}{=} \sum_{i=1}^{M} P(t,T_i) \left(\frac{P(t,T_{i-1})}{P(t,T_i)} - 1 \right) \\
&= \sum_{i=1}^{M} P(t,T_{i-1}) - P(t,T_i) \\
&= 1 - P(t,T_M)
\end{aligned}$$

□

In particular, (9) shows that NPV(t) does not depend on the tenor structure of the swap.

Definition 8. A **tenor basis swap** is a floating for floating IRS. Typically floating cash flows from two different Libor indices of the same currency are exchanged, e.g. 3m-Libor vs. 6m-Libor cash flows, which we denote by 3m-6m-Libor tenor basis swap. A so-called »tenor basis spread« s is added to the Libor index with the lower index to quote a tenor basis spread with a fixed maturity at par.

Definition 9. An **overnight (index) rate** is usually computed as a weighted average of overnight unsecured lending between large banks. Important examples are the effective Federal Funds Rate (FFR) for USD, the Euro OverNight Index Average (EONIA) and the Sterling OverNight Index Average (SONIA). In some countries, central banks publish a target overnight rate to influence monetary policy, for instance in the US. In contrast to Libor, it is **based on actual transactions** by definition.

Definition 10. An **overnight indexed swap (OIS)** is an interest rate swap where the floating payment is calculated via an overnight rate. The fixed rate of the OIS is typically an interest rate considered less risky than the corresponding Libor rate because it contains lower counterparty risk. Please note that the term »OIS rate« refers to the fixed rate of the OIS and *not* to the reference rate.

Remark. The **Libor-OIS spread** is seen as a measure for the health of banks since it reflects what banks believe is the risk of default when lending to another bank. Prior to 2007, the spread between the two rates used to be as little as 0.01%. A widening of the gap, as it was the case during the crisis, is a sign that the financial sector is stressed. In early 2018, at the time of writing this document, it was close to 0.6% – its highest level during the past ten years. However, the current spread is only observable in the US and analysts claim that its increase is not critical and exists only due to effects of recent fiscal policies.

Equations (5) and (9) imply that in the single-curve approach both Libor-OIS spreads and tenor basis spreads are always equal to zero, which can be verified empirically. However,

Fig. 2: 3m-Libor-OIS spread (1y); source: Bloomberg

Fig. 3: 3m-Libor vs. 6m-Libor tenor basis spread (5y); source: Bloomberg

the financial crisis of 2007–2008 has shown that this is not the case, as Fig. 2 and 3 indicate. For further visualisations see also [26].

We make two decisive observations that motivate the usage of the multi-curve approach, which will be introduced in the next chapter:

- In the second proof of Lemma 3.1 the FRA has tenor $\Delta_{FRA} = T_i - T_{i-1}$, whereas the bonds that are bought at time point t have tenor $\Delta_{Bonds} = T_i - t$ and therefore for $t \neq T_{i-1}$ we have
$$\Delta_{FRA} \neq \Delta_{Bonds} \,. \tag{10}$$

- On the other hand we know since the crisis that financial instruments with a longer tenor have larger liquidity and counterparty credit risks. These risks are defined from the viewpoint of a specific market participant A as follows:
  - **Market liquidity risk** is the risk that A will have difficulty selling an asset without incurring a loss. It is typically indicated by an abnormally wide bid-ask spread. It can be caused by A itself, if its position is large relative to the market, or exogenously by a reduction of buyers in the marketplace. In the subprime mortgage crisis, which initiated the financial crisis of 2007–2008, rapid endorsement and later abandonment of complicated structured financial instruments such as collateralised debt obligations (CDOs) lead to an immense drop in market prices and thereby to a loss of liquidity. Market liquidity risk is positively correlated to funding liquidity risk.
  - **Funding liquidity risk** is the risk that A will become unable to settle its obligations with immediacy over a specific time horizon and, as a result, will have to liquidate a position at a loss that it would keep otherwise. In the run up to the financial crisis, many banks were engaging in funding strategies that heavily relied on short-term funding thus significantly increasing their exposure to funding liquidity risk, see [21]. When banks such as Bear Stearns and Lehman Brothers started to look

vulnerable, their clients risked losing capital during a bankruptcy and they started to withdraw money and unwind positions, which lead to a bank run. This in turn increased market illiquidity with bid-ask spreads widening and as a consequence prices dropped.

- **Counterparty (credit) risk** or **default risk**, is the risk that a financial loss will be incurred if one of A's counterparties does not fulfil its contractual obligations in a timely manner. For instance, when Lehman Brothers filed for bankruptcy it was a counterparty to 930,000 derivative transactions which represented approximately 5% of global derivative transactions according to [18].

Since financial instruments with a longer tenor have larger liquidity and counterparty credit risks, it makes no sense that the NPV of a Libor swap's floating leg at time t does not depend on the tenor of the swap. However, this is being implied by Lemma 3.2.

One way to deal with the risk inconsistency mentioned in the second observation is to model these risks explicitly, so that the different rates become compatible with one another. Another way to tackle the problem is to segment market rates according to their tenor.

The second approach is suggested by Morini in [27], where he argues that an IRD with tenor Δ should only be replicated with IRDs of the same tenor Δ. Therefore, he implicitly does not recommend the procedure in the second proof of Lemma 3.1, as there we have in general $\Delta_{FRA} \neq \Delta_{Bonds}$, see (10).

Conclusion: The need to consider liquidity and counterparty credit risks when pricing IRDs is one of the key insights of the financial crisis of 2007–2008. This insight constitutes the decisive turning point of the pricing approach for IRDs.

4 Multi-Curve Approach: One Discount Curve and Distinct Forward Curves

The underlying idea of the multi-curve approach is to **segment market rates according to their tenor**. Thereby, we overcome the risk inconsistency discussed at the end of the last section. Before we illustrate this concept with some examples in Section 4.2, we first provide some historical and theoretical background to motivate and specify important details.

4.1 Background

4.1.1 Historical Background: New Regulations and the Rise of OIS

After the crisis, many regulatory steps were taken in order to address the solvency and liquidity problems that arose during the crisis. Important regulations are the Dodd-Frank Act and Basel III, which include provisions that tighten bank capital requirements, introduce leverage ratios and establish liquidity requirements.

Similarly, there has also been a higher attention on the counterparty credit risk. The following two key instruments attempt to reduce this risk:

- **Collateral agreements:** A collateral agreement is an additional contract to a main contract where the terms for the exchange of the collateral as a security are specified. There is a wide range of eligible collaterals which goes from cash to government or corporate bonds and more rarely bullions. If the NPV of the main contract is positive for A and exceeds a certain threshold by X, party A receives the collateral with value X from party B. As long as B is not in default it remains the owner of the collateral from an economic point of view and A needs to pass on coupon payments, dividends and any other cash flows to B. If the difference between the NPV and the value of the collateral position is in excess of the Minimum Transfer Amount (MTA), extra collateral needs to be posted. Collateralised transactions pose less counterparty risk because the collateral can be used to recoup any losses.

- **Central (clearing) counterparty (CCP):** In the aftermath of the crisis, authorities tried to push derivatives markets towards collateralisation of OTC transactions. The Dodd-Frank Act and the European Market Infrastructure Regulation (EMIR) intend to mitigate counterparty credit risk through the creation of CCPs. A CCP is a financial institution that interposes itself between counterparties of contracts, becoming the buyer to every seller and the seller to every buyer. It provides greater transparency of the risks, reduced processing costs and established processes in case of a member's default, see [34]. The most important aspect of central clearing is the **multilateral netting** of transactions between market participants, which simplifies outstanding exposures compared to a complex web of bilateral trades.

  We illustrate this effect with the simplest possible example: We have three market participants, A, B and C. A has to post a collateral of Y to B, B has to post a collateral of Y to C and C has to post a collateral of Y to A. If we consider the exact same situation only with a central counterparty in place, which is allowed to apply multilateral netting, then no party has to post a collateral any more. This is the case, since each of the three parties posts and receives the same amount of collateral.

  However, one should not forget that CCPs cannot fully eliminate counterparty credit risk. Furthermore, they concentrate risk, their probability of default is positive and they can be sources of financial shocks if they are not properly managed.

For transactions that are not centrally cleared by a CCP, regulators also impose the inclusion of a **Credit Support Annex (CSA)**, a document where the collateralisation terms and conditions are determined in detail. According to the International Swaps and Derivatives Association (ISDA), cash represents around 77% of collateral received and around 78% of collateral delivered against non-cleared derivatives in 2014, see [17]. The **collateral rate** which is being paid on cash collateral is also the effective funding rate for the derivative, as shown in [8]. This means that the appropriate rate to discount cash flows when valuing a collateralised trade corresponds to the collateral rate, which is in most cases an overnight rate. For instance, in the ISDA CSA for OTC derivative transactions, the collateral rate is usually determined as the OIS rate, i.e. the fixed rate of the OIS. Hence, we assume in the sequel that the collateral rate is the OIS rate.

OIS is the most prevalent choice amongst collateral rates because it is seen as the best estimate of the theoretical concept of the risk-free rate. A good approximation of the risk-free rate is desirable, since the collateral has effectively eliminated counterparty risk. As mentioned before, until the financial crisis Libor was also assumed to be a good such estimate, but during the crisis the Libor-OIS spread spiked to an all-time high of 3.64%. That Libor cannot be assumed to be risk-free was also discussed in the media in the aftermath of the Libor manipulation scandal of 2011, see [4] and Chapter 7. Whereas the overnight rates on which OIS are based are averages of *actual* transactions, Libor often just reflects the *opinion* of several banks at which rate other banks would let them borrow money, see Chapter 7.

4.1.2 Theoretical Background

Definition 4.1

With the risk neutral pricing approach, see [23], we obtain the **net present value (NPV)** at time t of a financial transaction with net cash flows $\text{Cash}(T_i)$ at time points T_i, $i \in \{1, \dots, M\}$, by

$$\text{NPV}(t) := \mathbb{E}_t \left[\sum_{i=1}^{M} D(t, T_i) \cdot \text{Cash}(T_i) \right],$$

where

- as before, $r(s)$ denotes the short rate at time s
- $D(t, T)$ denotes the discount factor

$$D(t, T) := \exp\left(- \int_t^T r(s) \mathrm{d}s \right)$$

- and the expectation is taken with respect to the **risk neutral pricing measure** and the filtration $\mathcal{F}_t$, which encodes the market information available up to time t.

Unfortunately, we do not know $D(t, T)$ at t. As suggested earlier in (1), there is a useful relation between $D(t, T)$ and the zero-coupon bond corresponding to $r(s)$

$$P(t, T) = \mathbb{E}_t[D(t, T)] = \mathbb{E}_t \left[\exp\left(- \int_t^T r(s) \mathrm{d}s \right) \right].$$

By a change of numeraire to $P(t, \cdot)$ we obtain

$$\text{NPV}(t) = \sum_{i=1}^{M} P(t, T_i) \mathbb{E}_t^i [\text{Cash}(T_i)] , \tag{11}$$

where $\mathbb{E}_t^i[\cdot]$ denotes the expectation under the T_i-forward measure associated with the risk-free zero-coupon bond $P(t,\cdot)$ and the filtration $\mathcal{F}_t$.[6]

We can now use (11) to calculate the NPV if we know to which interest rate the short rate $r(s)$ corresponds. In [32], it is shown that if the transaction is

(a) uncollateralised with no counterparty credit risk then $r(s)$ corresponds to the counterparty's funding rate.

(b) completely collateralised then $r(s)$ corresponds to the collateral rate.

The first case is of rather theoretical nature, since in practice, uncollateralised transactions usually involve counterparty credit risk. Note that if we are in the first case, then we would again use Libor in the interbank sector, just as in the single-curve approach.

We will assume in the following that we are in the second case, i.e. that our contracts are collateralised, as this is standard nowadays. For example, in 2014 around 97% of non-cleared credit derivatives and 91% of non-cleared equity derivatives were already using CSAs, see [17]. In this case it is reasonable to assume that the collateral rate is the OIS rate as, again, this is market standard, see [12].

4.2 Basic Concept and Important Examples

In conclusion, in the multi-curve approach we first build one single zero-bond curve from OIS rates. We continue to denote this zero-bond curve by $P(t,\cdot)$ and will explain its construction later in Section 4.3.1.

Apart from the zero-bond curve, we segment the interest rate market with respect to the different tenor structures of its derivatives. We use in each partial market a separate interest rate structure to value its IRDs. For instance, we use FRAs with 6m-tenor for the first two years and afterwards Libor swaps with 6m-tenor to account for the forward rates with 6m-tenor. This will be illustrated in Section 4.3.2.

Assumption: The IRD market should be segmented according to the tenor of its products.

Procedure: For pricing a linear IRD with tenor Δ, we

1. calculate the future cash flows with the **forward rate curve of tenor** Δ and
2. discount these future cash flows with a **unique zero-bond curve**.

Due to the importance of tenors in the multi-curve approach and for the sake of consistency, we alter the notation of the simply compounded spot rate $L(T-\Delta,T)$ of (3) to $L^{\Delta}(T-\Delta,T)$.

Definition 11. Consider a linear IRD with cash flows

$$\text{Cash}(T_i) \coloneqq \alpha_i + \beta_i \cdot L^{\Delta}(T_{i-1},T_i) ,$$

where $\alpha_i,\beta_i \in \mathbb{R}$ for $i = 1,\dots,M$ and $T_1 < \dots < T_M$ with $T_j - T_{j-1} = \Delta$ for

[6] In fact, this is how we defined the NPV earlier in Definition 7.

$j = 2, \dots, M$. With (11) we obtain

$$\begin{aligned}\mathrm{NPV}(t) &= \sum_{i=1}^{M} P(t, T_i) \cdot \mathbb{E}_t^i\Big[\mathrm{Cash}(T_i)\Big] \\ &= \sum_{i=1}^{M} P(t, T_i) \cdot \Big(\alpha_i + \beta_i \cdot \mathbb{E}_t^i\Big[L^\Delta(T_{i-1}, T_i)\Big]\Big) .\end{aligned} \tag{12}$$

We set $L^\Delta(t, T_{i-1}, T_i) := \mathbb{E}_t^i\Big[L^\Delta(T_{i-1}, T_i)\Big]$ and call the curve given by the mapping $T \mapsto L^\Delta(t, T - \Delta, T)$ the Δ-fixing curve at time t.

Remark. In the literature, one often defines the Δ-fixing curves using risky zero-bond curves $P^\Delta(t, \cdot)$ by

$$L^\Delta(t, T_{i-1}, T_i) := \frac{1}{\tau(T_{i-1}, T_i)} \left(\frac{P^\Delta(t, T_{i-1})}{P^\Delta(t, T_i)} - 1 \right) .$$

This is motivated by (5). We then have all curves of interest, i.e. the zero-bond curve and the different forward curves, in »zero-bond form«. Here in this white paper, we omit this practice as it would not add any extra insights.

For valuing an IRD as in the above definition we need to know the specific Δ-fixing curve. Therefore, we would like to discuss the value of a FRA:

Example 2. For the standard FRA, which we have introduced in Definition 6, the payment date is assumed to be T_i, so at time T_i we have the payoff

$$FRA_{std} := FRA_{std}(t, T_{i-1}, T_i, N) := N\tau(T_{i-1}, T_i)\Big(K - L^\Delta(T_{i-1}, T_i)\Big) ,$$

where K is the FRA rate, N the nominal value and $\Delta = T_i - T_{i-1}$. With (11), the NPV at time t is given by

$$\begin{aligned}NPV_{FRA_{std}} &= P(t, T_i) \cdot \mathbb{E}_t^i\Big[N\tau(T_{i-1}, T_i)\Big(K - L^\Delta(T_{i-1}, T_i)\Big)\Big] \\ &= P(t, T_i) \cdot N\tau(T_{i-1}, T_i)\Big(K - \mathbb{E}_t^i\Big[L^\Delta(T_{i-1}, T_i)\Big]\Big) .\end{aligned}$$

Since K is chosen such that $NPV_{FRA_{std}} = 0$ we get

$$L^\Delta(t, T_{i-1}, T_i) := \mathbb{E}_t^i\Big[L^\Delta(T_{i-1}, T_i)\Big] = K . \tag{13}$$

Example 3. The actual FRA traded on the market, however, has payment date T_{i-1} and is discounted with the floating spot rate, i.e. at time T_{i-1} we have the payoff

$$FRA_{mkt} := FRA_{mkt}(t, T_{i-1}, T_i, N) := N\tau(T_{i-1}, T_i)\frac{K - L^\Delta(T_{i-1}, T_i)}{1 + \tau(T_{i-1}, T_i)L^\Delta(T_{i-1}, T_i)} .$$

Note that the discounting with the floating spot rate is not a modelling choice but specified in the contract itself. Mercurio shows in [25] that the NPV at time t is given by

$$NPV_{FRA_{mkt}} = P(t, T_{i-1})N\left(\frac{1 + K \cdot \tau(T_{i-1}, T_i)}{1 + L^\Delta(t, T_{i-1}, T_i)\tau(T_{i-1}, T_i)} \exp(C(t, T_{i-1})) - 1\right) ,$$

where $\exp(C(t, T_{i-1}))$ is called »convexity adjustment« and depends on the model. He further proves that under reasonable model assumptions and if the difference between forward Libor rates and corresponding OIS rates remains fairly constant, which is usually the case, then the value of $C(t, T_{i-1})$ is negligibly small and we further have

$$NPV_{FRA_{mkt}} \approx NPV_{FRA_{std}} = P(t, T_i) \cdot N\tau(T_{i-1}, T_i)\Big(K - L^\Delta(t, T_{i-1}, T_i)\Big) , \tag{14}$$

which again results in $L^\Delta(t, T_{i-1}, T_i) = K$.

Remark. Note that FRA contracts are quoted on the market in terms of the FRA equilibrium rates. They are also included into rates of futures, interest rate swaps and tenor basis swaps. Therefore, the different FRA curves can be extracted from market quotations.

Example 4. To illustrate the use of Definition 11 we consider a 6m-Libor fixed for floating swap over the term of two years. We will use the resulting pricing formula to construct the 6m-fixing curve in Section 4.3.2. For simplicity, we drop the usage of day count conventions at this point and assume that the payments take place semi-annually. We recommend to compare the following procedure to the one used in the first proof of Lemma 3.1, where we applied the single-curve approach to price a standard FRA.

- The fixed party pays at time $i \cdot 6m$ the cash flows

$$\text{Cash}(i \cdot 6m) = S(0, 2y) \cdot 6m = \alpha_i + \beta_i \cdot L^{6m}((i-1) \cdot 6m, i \cdot 6m) ,$$

  with the swap fixed rate $S(0, 2y)$, $\alpha_i = S(0, 2y) \cdot 6m$ and $\beta_i = 0$. With (12) we get the fixed leg

$$\text{NPV}(0) = S(0, 2y) \cdot 6m \sum_{i=1}^{4} P(0, i \cdot 6m) .$$

- The variable party pays at time $i \cdot 6m$ the cash flows

$$\text{Cash}(i \cdot 6m) = L^{6m}((i-1) \cdot 6m, i \cdot 6m) \cdot 6m = \alpha_i + \beta_i \cdot L^{6m}(T_{i-1}, T_i) ,$$

  with $\alpha_i = 0$ and $\beta_i = 6m$. With (12) and the 6m-fixing curve we get the floating leg

$$\text{NPV}(0) = \sum_{i=1}^{4} P(0, i \cdot 6m) \cdot 6m \cdot L^{6m}(0, (i-1) \cdot 6m, i \cdot 6m) .$$

- So, in summary we get

$$S(0, 2y) = \frac{\sum_{i=1}^{4} P(0, i \cdot 6m) \cdot L^{6m}(0, (i-1) \cdot 6m, i \cdot 6m)}{\sum_{i=1}^{4} P(0, i \cdot 6m)} .$$

In general, i.e. with day count conventions and different fixed and floating payment dates, we obtain

$$S(0, T_n) = \frac{\sum_{i=1}^{\tilde{n}} P(0, \tilde{T}_i) \tau(\tilde{T}_{i-1}, \tilde{T}_i) L^{\Delta}(0, \tilde{T}_{i-1}, \tilde{T}_i)}{\sum_{i=1}^{n} \tau(T_{i-1}, T_i) P(0, T_i)} , \tag{15}$$

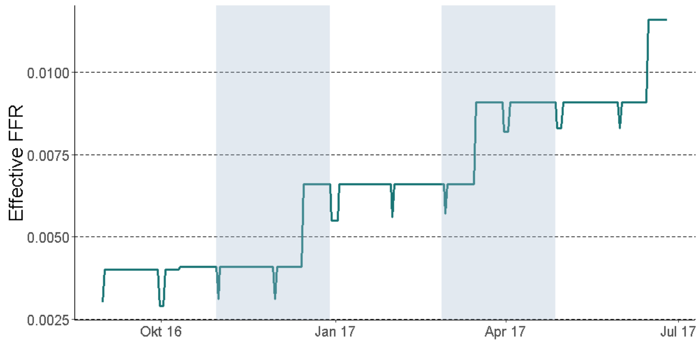
where $\mathcal{T} = \{T_1, \dots, T_n\}$ is the time structure of the fixed cash flows and $\tilde{\mathcal{T}}^{\Delta} = \{\tilde{T}_1^{\Delta}, \dots, \tilde{T}_{\tilde{n}}^{\Delta} = T_n\}$ is the time structure of the floating cash flows with $\tilde{T}_{i+1}^{\Delta} = \tilde{T}_i^{\Delta} + \Delta$, $i = 1, \dots, \tilde{n} - 1$.

4.3 Curve Construction

We start with the construction of the zero-bond curve from OIS rates in Section 4.3.1, as this curve will be used to construct the forward curves in Section 4.3.2.

4.3.1 OIS Curve Bootstrapping

In the sequel we follow [20] and [9]. The starting point of the multi-curve approach is always the construction of a zero-bond curve $P(t, \cdot)$. We denote by



Multi-Curve Approach: One Discount Curve and Distinct Forward Curves

Fig. 4: Daily effective Federal Funds Rate; source: St. Louis Fed

- N the number of business days in a given period
- τ_i the year fraction between the business day i and the next business day, for example we normally have for i being a Friday $\tau_i = 3/$(number of days in the specific year)
- REF$_i$ the reference rate published for business day i which is valid until the next business day and usually published on business day $i+1$.

The paid interest over this period is

$$\prod_{i=1}^{N}(1+\tau_i \cdot \text{REF}_i) - 1 \, .$$

Note that the final settlement of an OIS occurs one day after the maturity date of the OIS due to the delay of publishing the reference rate for the maturity date, see [29].

A special feature of OIS rates is the quasi-static behaviour of reference rates between Monetary Policy Meeting Dates. Another speciality is that there are often seasonality effects observable at each quarter or end of the year, see Fig. 4.

We assume that the seasonality adjustment is built into the rates REF$_i$ by adding a spread s_i which can be obtained through historical data or through estimations. By

$$\mathcal{T} := \mathcal{T}_{\text{short}} := \{0 = t_0, t_1, \ldots, t_n\} \quad \text{and} \quad r_{\mathcal{T}} := \{r_{t_1}, \ldots, r_{t_n}\}$$

we denote critical dates of the short part of the curve, such as meeting dates and regular tenor OIS dates, and their corresponding quoted rates for these periods. We further assume that these forward starting OIS rates are quoted from the spot date to the meeting date. Clarke suggests in [8] that this short term period lasts from 3 to 6 months. Our aim is to determine the daily rates REF$_i$. The identity

$$r_{t_1} \cdot \tau(t_0, t_1) = \prod_{i=1}^{N_{t_0,t_1}} \Big(1 + \tau_i \cdot (\text{REF}_i + s_i)\Big) - 1$$

reveals the relation of the rates $\tilde{r}_{t_i}$ and r_i, where we have the fixed leg on the left hand side and the floating leg on the right hand side and where N_{t_{i-1},t_i} denotes the number of business days in the period from t_{i-1} to t_i. Because of the quasi-static behaviour of the rates r_i between meeting days we only consider constant rates $r_{i,i+1}$ between day i and $i+1$ and get

$$r_{t_1} \cdot \tau(t_0, t_1) = \prod_{i=1}^{N_{t_0,t_1}} \Big(1 + \tau_i \cdot (\text{REF}_{t_0,t_1} + s_i)\Big) - 1 \, .$$

Hence, we can now solve for REF$_{t_0,t_1}$. With the calculated REF$_{t_0,t_1}$ we then obtain

REF_{t_1,t_2} through

$$r_{t_2} \cdot \tau(t_0, t_1) = \prod_{i=1}^{N_{t_0,t_1}} \Big(1 + \tau_i \cdot (\text{REF}_{t_0,t_1} + s_i)\Big) \cdot \prod_{i=t_1+1}^{N_{t_1,t_2}} \Big(1 + \tau_i \cdot (\text{REF}_{t_1,t_2} + s_i)\Big) - 1$$

and stress that $i = t_1 + 1$ refers to the first business day after business day t_1.

This way we can obtain all rates $\text{REF}_{t_i,t_{i+1}}$.

In the middle part of the curve, from the end t_n of the short part up to one year, normal interpolation of OIS rates can be used. Another possibility is to continue using the daily discount process as done in the short dated part, but this only has minimal benefits, as stated in [9].

The long region starts after one year, where OIS pays annual interest. Here we can again apply the classical single-curve bootstrapping technique for $1 < T$ as described in Section 3.1, since this part can be assumed to show no step function behaviour due to the greater uncertainty of the responsible committees' actions over longer horizons.

If we need a zero-bond curve for values t greater than the maturity of the longest OIS we could

(a) assume that the spread between the OIS fixed rates and the Libor swap fixed rates is constant for all maturities after the longest OIS maturity or

(b) use basis swaps where 3m-Libor is exchanged for the average OIS reference rate plus a spread.

It is important to also acknowledge the step function characteristics of the short-dated region with a suitable interpolation scheme and to choose the interpolation scheme of preference in the subsequent part.

We mentioned earlier that at each quarter or the end of the year, extreme seasonality effects are observable. In the literature this is commonly referred to as the »turn-of-the-year (TOY) effect«. The bootstrapping can be sensitive to this effect and it is recommended to first exclude it from the data and to model it on top of the bootstrapped curve after the bootstrapping has been completed. We refer to [13], where Ametrano and Bianchetti have outlined such a possible approach.

4.3.2 Forward Curves Bootstrapping

In the multi-curve approach, the unknown forward rate curves for the different tenors must be bootstrapped relative to the known zero-bond curve $P(t, \cdot)$, which was priorly bootstrapped using OIS rates. In the following, we denote the forward rate curve for the tenor Δ by Δ-curve. In the single currency setting, we are usually interested in the standard tenors 1m, 3m, 6m and 12m. If we have set up these curves there exist techniques to obtain, for instance, the 2m-curve, by interpolating between the 1m-and the 3m-curve. However, we will not cover them in this white paper.

The first step in constructing forward curves is a careful selection of the corresponding bootstrapping instruments. Contemplable instruments may overlap one another in some areas. Hence, we select those that are least overlapping and give preference to the more liquid ones, i.e. those with tighter bid-ask spreads.

The reference date for all the EUR market bootstrapping instruments – except for overnight and tomorrow-next deposit contracts – is T_0, with T_0 = spot date. Once the Δ-curve at

T_0 is available, the corresponding Δ-curve at t_0, with t_0 = today, can be obtained using the discount factor between these two dates implied by overnight and tomorrow-next deposits.

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We discuss here the construction of the 6m-curve, because this segment is the most liquid in the EUR market. For this purpose, we use the instruments specified in Fig. 5.[7]

Index	Type	Duration
1	FRA rate	0x6
2	FRA rate	1x7
3	FRA rate	2x8
$\vdots$	$\vdots$	
19	FRA rate	18x24
20	6m-Swap fixed rate	3 years
21	6m-Swap fixed rate	4 years
$\vdots$	$\vdots$	
46	6m-Swap fixed rate	29 years
47	6m-Swap fixed rate	30 years

Fig. 5: Input rates for bootstrapping the 6m-curve after the financial crisis

We have already derived the valuation formulas for FRAs and swaps, given by (14) and (15), respectively. With these we will now, similar to Section 3.1, bootstrap grid points of the fixing curve $L^{6m}(t, T-6m, T)$ from the above N rates. As before, different calculations apply:

- For $T \leq 2y$:

  As we have in (13)

  $$L^{\Delta}(t, T_{i-1}, T_i) := \mathbb{E}_t^i\Big[L^{\Delta}(T_{i-1}, T_i)\Big] = K \ ,$$

  where K is the corresponding FRA rate, we can just read the values for $L^{\Delta}(t, T_{i-1}, T_i)$ »off the screen«.

- For $T \geq 3y$:

  In the case of swaps we obtain from (15)

  $$\begin{aligned} L^{6m}(0, \tilde{T}_{\tilde{n}-1}, \tilde{T}_{\tilde{n}}) = & \frac{S(0, T_n)\sum_{i=1}^{n}\tau(T_{i-1}, T_i)P(0, T_i)}{P(0, \tilde{T}_{\tilde{n}})\tau(\tilde{T}_{\tilde{n}-1}, \tilde{T}_{\tilde{n}})} \\ & - \frac{\sum_{i=1}^{\tilde{n}-1} P(0, \tilde{T}_i)\tau(\tilde{T}_{i-1}, \tilde{T}_i)L^{6m}(0, \tilde{T}_{i-1}, \tilde{T}_i)}{P(0, \tilde{T}_{\tilde{n}})\tau(\tilde{T}_{\tilde{n}-1}, \tilde{T}_{\tilde{n}})} \ , \end{aligned}$$

  where $\mathcal{T} = \{T_1, \ldots, T_n\}$ is the time structure of the fixed cash flows and $\tilde{\mathcal{T}}^{6m} = \{\tilde{T}_1^{6m}, \ldots, \tilde{T}_{\tilde{n}}^{6m} = T_n\}$ is the time structure of the floating cash flows with $\tilde{T}_{i+1}^{6m} = \tilde{T}_i^{6m} + 6m$, $i = 1, \ldots, \tilde{n}-1$ and $n = 3y, \ldots, 30y$.

  In practice, since the market's fixed leg frequency is annual and the floating leg frequency is given by the underlying Libor rate tenor $\Delta \leq 1y$, we might have to use interpolation during the bootstrap procedure, compare the end of Section 3.1.

[7] Note that we are in general not allowed to simply use 6m-deposits, as they are neither Libor-indexed nor collateralised.

For setting up the other curves usually the following instruments are used:

- 1m-curve: Monthly swaps and basis swap quotes against other swap rates
- 3m-curve: FRA rates, futures and basis swap quotes against 6m-swap rates
- 12m-curve: 12x24 FRA and basis swap quotes against the 3m-curve or 6m-curve

In general, one possibility for calculating the Δ-curve is to add the corresponding tenor basis spread curve, i.e. the OIS-Δ-spread curve, to the zero-bond curve obtained in the last section. If this curve is not available one can first add the OIS-x-spread curve, with the highest possible $x \leq \Delta$ and then add the x-y-spread curve with the highest possible y such that $x + y \leq \Delta$ and so on ... These constructed instruments are the so-called »synthetic deposits« and are explained in more detail in [31].

After having constructed the different Libor forward rates, we can again get the whole curve of interest by applying interpolation methods, compare the end of Section 3.1. Using interpolation techniques for estimating the different Δ-curves is also known as the »**exact fit approach**«.

Instead of the exact fit approach, which usually results in an overfitted curve, one can also apply a so-called »**best fit approach**«, which returns a more realistic curve. The most popular examples of best fit approaches are the **Nelson-Siegel model** and its extension, the **Svensson model**. Best fit approaches are used by most central banks, particularly because their estimations are better suited for international comparison compared to the usage of spline interpolation, as they make use of parameters and spline interpolation does not. The Svensson model involves six parameters – two more than the Nelson-Siegel model – which allows for a second hump in the curve. We want to find parameters such that the theoretical prices are as close as possible to the bootstrapped prices. For this purpose the ordinary least squares method is typically used. In summary, both models are not extremely complex, fit the data well and also work with a small amount of data points, see [33] for more information and an introduction to the R package *termstrc*.

Remark. Note that in general we have

$$L^{\Delta}(T_{n-1}, T_n) \neq L^{\Delta}(T_{n-1}, T_{n-1}, T_n) \;,$$

compare (7) in the single-curve approach.

See [13] for the pricing formulas of more financial instruments, such as basis swaps and futures, and for further details. They used and recommend the open-source QuantLib framework to obtain numerical results.

4.4 Validation of the Constructed Curves

Before applying the constructed curves to real problems we should always carry out some validation. As pointed out in [20], there are four major steps that should be carried out:

1. Check if
   (a) in case of an exact fit approach, the built curves correctly reprice the market instruments that were used as an input for the curve construction.
   (b) in case of a best fit approach, the built curves reprice the market instruments that were used as an input for the curve construction with a predetermined precision.

2. Check if the built curves correctly price the market instruments that were not used as an input for the curve construction.

3. Check if the market rates with which the trading desk works are recovered exactly.

4. Check if the generated forward rates are sufficiently smooth. Jumps could be caused by the usage of a non-optimal interpolation scheme during the bootstrapping approach itself.

Likewise, the authors of [13] recommend that any good bootstrapping system should provide a real-time snapshot of yield curve shapes and a real-time pricer of market instruments for each constructed curve. As a side note, they also show that our bootstrapping framework is robust enough to be able to deal with negative market rates.

5 Generalisation: Collateral in a Foreign Currency

Some collateral agreements restrict the number of currencies in which the collateral can be posted. As explained in the previous section, collateralised trades should be discounted using the collateral rate, no matter what the currency of the underlying transaction is. In [22], the authors have found that the »cheapest-to deliver« option can significantly change the fair value of a trade when a contract allows multiple currencies as eligible collateral as well as its free replacement. Hence, it is important to have a closer look at this important issue.

From now onwards we follow [5], where it is shown how zero-coupon bond prices in a foreign currency USD, denoted by $P^{\text{USD}}(t,T)$, can be obtained through the zero-coupon bond prices in our local currency EUR, denoted by $P(t,T)$. The resulting curve $P^{\text{USD}}(t,\cdot)$ can then be used to value payments in USD in a contract collateralised in EUR.

We enter

1. a cross-currency swap (CCS) and therefore exchange at time $T_i, i \in \{1,\dots,N\}$, floating payments plus a basis spread bs_i in our local currency EUR for floating payments in the foreign currency USD. It holds that $\Delta = T_i - T_{i-1}, i = 2,\dots,N$.

2. a fixed for floating interest rate swap in the foreign currency to exchange exactly the floating payments in the foreign currency for fixed payments in the foreign currency. This we do, to be left with only one unknown variable, namely the foreign discount factors $P^{\text{USD}}(t,T)$.

Assuming coinciding payment dates and day count conventions for the two swaps, the following system of equations has to be fulfilled:

$$
\begin{aligned}
P(0,T_i) &+ \sum_{i=1}^{n} P(0,T_i)(L^{\Delta}(0,T_{i-1},T_i) + bs_i) \cdot \tau(T_{i-1},T_i) \\
&= P^{\text{USD}}(0,T_n) + c_n \cdot \sum_{i=1}^{n} P^{\text{USD}}(0,T_i) \cdot \tau(T_{i-1},T_i) \ ,
\end{aligned}
$$

for $1 \le n \le N$, where c_n denotes the par spread of the USD-IRS. The left hand side of the equation is the part that we pay, whereas the right hand side is the part that we receive. By bootstrapping these equations we get

$$
\begin{aligned}
P^{\text{USD}}(0,T_n) = \frac{P(0,T_n) + \sum_{i=1}^{n} P(0,T_i)(L^{\Delta}(0,T_{i-1},T_i) + bs_i) \cdot \tau(T_{i-1},T_i)}{1 + c_n \cdot \tau(T_{n-1},T_n)} \\
- \frac{c_n \cdot \sum_{i=1}^{n-1} P^{\text{USD}}(0,T_i) \cdot \tau(T_{i-1},T_i)}{1 + c_n \cdot \tau(T_{n-1},T_n)} \ .
\end{aligned}
$$

Remark. Note that we implicitly assumed that the CCS and the IRS that we enter are both also collateralised in our local currency EUR. However, this is usually not the case but seems unavoidable, as pointed out in [5].

6 Overview: Extensions of Interest Rate Models to the Multi-Curve World

So far, we have seen the progress from the single-curve approach to the multi-curve approach in the aftermath of the financial crisis. In the most general new framework we have different zero-bond curves for different collateral currencies and collateral rates and different forward Libor curves corresponding to different market tenors. This suffices to price linear IRDs, such as FRAs, futures and IRSs.

For **non-linear IRDs** that have more complex payouts and whose price has to be calculated by simulation, such as bond options, caplets and swaptions, it is necessary to have a dynamic model of the interest rate curve, just like in the single-curve world. There exist two types of models:

- Models of **reduced form**, where credit and liquidity risks are modelled.
- Models of **structured form**, where the interest rates and relevant spreads are modelled *directly*.

In practice, models of the latter type are used. These are also referred to as »stochastic spread models«. The downside of them is, however, that they do not help us to understand the reasons for the resulting spreads.

Until the financial crisis, there were three main families of models for interest rate dynamics, each providing a different solution based on different choices of the underlying modelling variables. In the following, we refer to [24] and provide a brief overview of these different families and link to relevant papers that have extended each of them to the multi-curve world:

1. **Short-rate models:** This family of models was first introduced in 1977 by Vasicek and only models – as the name implies – the instantaneous short rate. In the single-curve context, all interest rates can be seen as functions of bonds and every bond can be interpreted as the expectation of a specific discount factor, which is itself defined as an integral over the short rate, see Definition 1. After establishing a numerical representation for the short rate we get analytical solutions for bond prices and can price any IRD of interest.

   Extension: In [19], Kenyon extends short-rate models to the multi-curve world. This class of models can only be used for pricing IRDs that can be described by bonds. It is shown that many useful analytical results can be obtained in this setting, such as swaption pricing and fitting the smile.

2. **Heath-Jarrow-Morton (HJM) models:** Introduced by Heath, Jarrow and Morton in 1992, this family models instantaneous forward rates which have perfect correlation along the yield curve. No drift estimation is needed, since drifts of the no-arbitrage evolution of certain variables are expressed as functions of their volatilities and the correlations among themselves. While short-rate models only model the dynamics of a point on the forward rate curve, namely the short rate, HJM-type models capture the full dynamics of the entire curve.

   Extension: In [28], Moreni and Pallavicini recognise that an extension of the HJM-family has the same limitations as in the case of short-rate models. Hence, they model discrete tenor forward rates instead of the instantaneous rates. In [35], Torrealba models different variables and uses theoretical bonds to describe rates. Thus, these two approaches treat the same problems in a fundamentally different way.

3. **Libor market models:** With these models real-world discrete-tenor rates are modelled instead of instantaneous variables, as the above two model families do. More specifically, they model forward rates, which are directly observable in the market and whose volatilities are directly linked to traded contracts. It is especially useful for very complex IRDs, such as autocaps, Bermudan swaptions, constant maturity swaps and zero coupon swaptions. In fact, we have already seen the link of these rates to bonds in Lemma 3.1.

   Extension: This family was the first to be extended to the multi-curve market by Mercurio in 2009. Also in this case, we already came across a central theoretical result in Example 2, namely that a forward rate can be written as the expectation of the future Libor rate paid by the FRA under the measure associated with the risk-free discounting bond, which was shown by Mercurio in [25]. His approach guarantees that FRA rates are higher than risk-free rates by first modelling the risk-free rates and then modelling the non-negative spread of them to the FRA rates.

   In more detail:

   - Consider a time structure $\mathcal{T}^\Delta = \{T_0^\Delta, \dots, T_M^\Delta\}$ and use the OIS bonds to define the OIS forward rate

     $$F_{OIS}^\Delta(t, T_{i-1}, T_i) = \frac{1}{\tau(T_{i-1}, T_i)} \left(\frac{P(t, T_{i-1})}{P(t, T_i)} - 1 \right),$$

     compare Lemma 3.1.

   - The spread $S^\Delta(t, T_{i-1}, T_i)$ between the FRA rate $L^\Delta(t, T_{i-1}, T_i)$ and the OIS forward rate $F_{OIS}^\Delta(t, T_{i-1}, T_i)$ is given by

     $$S^\Delta(t, T_{i-1}, T_i) \coloneqq L^\Delta(t, T_{i-1}, T_i) - F_{OIS}^\Delta(t, T_{i-1}, T_i)\,.$$

     Since by definition, the FRA rates and OIS forward rates are martingales with respect to the measure associated with the risk-free discounting bond, the same is true for their difference.

   - Model the joint evolution of $F_{OIS}^\Delta(t, T_{i-1}, T_i)$ and the non-negative spread $S^\Delta(t, T_{i-1}, T_i)$.

   In [1], Mercurio's extension is discussed in the case of the Heston stochastic volatility model with displaced diffusion. The paper particularly covers different approaches to swaption pricing based on different modelling approaches and provides details on calibration and delta hedging.

7 A Note on Libor: Its Rise, Scandal, Fall and Replacement

The London Interbank Offered Rate (Libor) is the trimmed average of interest rates estimated by each of the leading banks in London that they would be charged if they had to borrow unsecured funds in reasonable market size from one another. Libor rates were officially fixed for the first time on 1st January 1986 by the British Bankers' Association (BBA), an industry trade group, and were assumed to be approximately riskless until the financial crisis. The BBA intended to create a **uniform benchmark for banks** to wipe out the necessity of constantly haggling over the interest rates that would be charged for different types of loans. The key concept is that Libor is based upon the offered rate and not the bid rate, i.e. submissions are based upon the lowest perceived rate that a bank *could* – not *would* – go into the interbank money market and obtain funding, for a specific currency and maturity. Today, it is calculated daily for five currencies (CHF, EUR, GBP, JPY, USD), each having seven different tenors (1d, 1w, 1m, 2m, 3m, 6m, 12m). As a consequence, the rates are **not necessarily based on actual transactions**, since not all banks require funds in marketable size each day in each of the above currencies and maturities.

It has become a **foundation of global finance**, being the primary worldwide benchmark for short-term interest rates and hence is sometimes referred to as the »world's most important number«. Many credit card contracts, financial derivatives, mortgages, student loans and other financial products rely on Libor as a reference rate and it thereby affects consumers and financial markets worldwide. Since most derivatives are not traded on public exchanges, it is hard to say exactly how vast Libor's reach is, but it is estimated that at least USD 350 trillion of outstanding financial contracts worldwide are based on the benchmark, see [16]. This is nearly five times the value of the gross world product of 2016, see [36].

Similar reference rates set by the private sector are, for instance, the Euro Interbank Offered Rate (Euribor), the Singapore Interbank Offered Rate (Sibor) and the Tokyo Interbank Offered Rate (Tibor).

During the **Libor scandal of 2011** it was investigated that the rates have been consciously manipulated by many of the contributing banks. The two main motivations for a bank to submit inaccurate rates were essentially:

1. Making the bank look healthier than it really was during the financial crisis.

2. Benefiting the financial positions of the bank's traders that bet on the day's benchmark and thereby unfairly raising the bank's profits.

It turned out that similar benchmarks like Libor, including Euribor, Sibor and Tibor, have also been manipulated. The damage for the global economy was estimated to be in the billions.

In the aftermath, several significant reforms were introduced to the affected rates. In the case of Libor, these were:

- In early 2014, a new administrator, Intercontinental Exchange (ICE), took over.

- The number of rates were reduced from 150 to 35 and the calculation was altered so that they are more robustly based on underlying transactions.

- New UK laws have been passed that criminalise the manipulation of relevant benchmarks and brought Libor under UK regulatory oversight by the Financial Conduct Authority (FCA).

- Maybe most importantly, individual bank's submissions to Libor are now being published after a period of three months, so that there is no incentive to make inaccurate statements with the purpose to appear overly healthy. The submissions are of course still available in real-time to the administrators and regulators to calculate the daily rates and for surveillance purposes.

In 2014, Zimmermann welcomes these changes but recommends in Chapter 7 of [10] to examine the option of replacing private benchmark rates like Libor and Euribor by alternative benchmarks based on central bank key rates and emphasizes that the reform undertaken due to the Libor scandal

> »falls short of resolving the fundamental conflict of interests that arises from a situation in which staff members, who are all on the payroll of the same institution, and who see each other over lunch and who play golf with each other, are in charge both of contributing to the setting of a transparent and objective benchmark, thus providing a public service to the financial system, and of making money by betting on the evolution of the very same interest rates. In hindsight, the only thing that really is surprising in the entire Libor scandal is that anybody ever believed that the existing system would not be rigged.«

On 27th July 2017, Andrew Bailey, CEO of the FCA, announced the **end of Libor**. It is going to be phased out in all currencies and tenors by the end of 2021, see [2]. The main reason for this radical step is that the unsecured lending market on which Libor is based is no longer sufficiently active. Bailey mentioned one example, where there were only fifteen transactions in the whole of 2016 for a specific currency and lending period. This means that on most days the rate is set based on expert opinion alone. Until 2021, alternative interest rate benchmarks have to be found and a strategy for a smooth transition has to be worked out. Fortunately, all current panel banks agreed voluntarily to continue contributing to Libor during this period.

One thing is certain: The **new benchmarks** that will be chosen to replace Libor will be calculated based on actual transactions and will thereby reflect actual, not theoretical, borrowing costs. It is possible that algorithms will be involved in the computations in accordance with the increasingly computerised and robotic nature of the financial system. The FCA has suggested that a reformed SONIA could be considered as an alternative. Likewise in Switzerland, the National Working Group on Swiss Franc Reference Rates has proposed the Swiss Average Rate Overnight (SARON). The European Central Bank (ECB) has also announced the plan to create a new benchmark. In early April of 2018, the New York Fed launched the Secured Overnight Financing Rate (SOFR) set at 1.80 percent, which potentially is supposed to replace Libor. SOFR is based on the overnight treasury repurchase agreement market, which trades around USD 800 billion in volume daily.

After suitable benchmarks have been found, the important question arises how to deal with legacy financial contracts whose settlements are linked to Libor. It is possible, but not certain at all, that ICE will continue to issue Libor. In [11], an auction-and-protocol process is discussed to convert a Libor-based contract to another reference rate.

There is plenty of work to be done, yet we shall never forget:

> »*Success is a journey, not a destination.*«
>
> — Arthur Ashe (1943–1993)

A Appendix

A.1 Conventions

Throughout this document, the following conventions hold:

- We assume $t \leq T$.
- We assume that the considered IRDs only have one underlying interest rate and that this rate is Libor.
- We assume the Actual/360 day count convention. It is determined by the factor of the day count function $\tau(t,T) := \frac{1}{360}$Days($t$, T), i.e. one year is assumed to consist of 360 days. For further examples of day count functions see [29].

  Day count functions are typically monotonically increasing with increasing time intervals and additive, i.e.

  $$\tau(T_1,T_2) + \tau(T_2,T_3) = \tau(T_1,T_3) , \qquad T_1 \leq T_2 \leq T_3 .$$

- We assume that the collateral rate of a collateralised contract is an OIS rate.

A.2 Notation

Notation	Meaning
$D(t,T)$	Discount factor at time t for time period $[t,T]$
$L(t,T) = L^{\Delta}(t,T)$	Libor rate at time t for time period $[t,T]$, where $\Delta = T - t$
$L(t,T_{i-1},T_i)$	FRA rate at time t for time period $[T_{i-1},T_i]$
$L^{\Delta}(t,\cdot-\Delta,\cdot)$	Δ-fixing curve at time t
$P(t,T)$	Zero-coupon bond price at time t for time period $[t,T]$
$S(t,T)$	Swap fixed rate at time t for time period $[t,T]$
$\tau(t,\cdot)$	Day count function at time t (see annotation above)

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